Characteristic Vectors

If V is a finite-dimensional vector space and T is a linear transformation from V into V, then, as was shown in Section 1, T can be represented, in terms of coordinates with respect to a given basis, as multiplication by a matrix A. The choice of a different basis results in a different matrix B, which is similar to A. In the next two sections, we shall discuss the problem of finding, if possible, a basis such that the matrix B is diagonal. In this section a concept is presented that will be useful in those later discussions.

We say that a vector $\bar{v}$ is a **characteristic vector** for T and that λ is a **characteristic value** of T if

$$T\bar{v} = \lambda\bar{v} \quad \text{and} \quad \bar{v} \neq \bar{0}$$

In other words, a nonzero vector $\bar{v}$ is a characteristic vector if $T\bar{v}$ is a *multiple* of $\bar{v}$.

This concept gives a useful formulation of the problem of finding a diagonal matrix representation for T, for

1 Suppose T is a linear transformation from V into V and that the matrix B of T with respect to the basis $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ is diagonal. Then each $\bar{u}_i$ is a characteristic vector for T, and the diagonal entries of B are the corresponding characteristic values.

To simplify the proof of this result assume that $n = 2$. The columns of the matrix B are the coordinates of $T\bar{u}_1$ and $T\bar{u}_2$ with respect to the basis $\bar{u}_1, \bar{u}_2$. (See formula **6**, page 239.) Therefore

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad \text{where} \quad \begin{aligned} T\bar{u}_1 &= B_{11}\bar{u}_1 + B_{21}\bar{u}_2 \\ T\bar{u}_2 &= B_{12}\bar{u}_1 + B_{22}\bar{u}_2 \end{aligned}$$

Thus, if B is diagonal, then $B_{12} = B_{21} = 0$, and therefore

2 $$T\bar{u}_1 = B_{11}\bar{u}_1 \qquad \text{and} \qquad T\bar{u}_2 = B_{22}\bar{u}_2$$

The vectors $\bar{u}_1$ and $\bar{u}_2$ are independent and hence nonzero. We therefore conclude that $\bar{u}_1$ and $\bar{u}_2$ are characteristic vectors, with corresponding characteristic values B_{11} and B_{22}, the diagonal entries of B.

The converse of statement **1** is also true, because

3 If V has a basis $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$ consisting of characteristic vectors for T, the matrix of T with respect to this basis is diagonal.

To simplify the proof of this, assume that $n = 2$. Suppose $\bar{u}_1$ and $\bar{u}_2$ are a basis for V consisting of characteristic vectors for T. Therefore $T\bar{u}_1$ is a multiple of $\bar{u}_1$ and $T\bar{u}_2$ is a multiple of $\bar{u}_2$, so we can write

$$T\bar{u}_1 = a\bar{u}_1 \qquad \text{and} \qquad T\bar{u}_2 = b\bar{u}_2$$

Rewrite this in the form

$$T\bar{u}_1 = a\bar{u}_1 + 0\bar{u}_2 \qquad \text{and} \qquad T\bar{u}_2 = 0\bar{u}_1 + b\bar{u}_2$$

Thus the matrix of T with respect to $\bar{u}_1$ and $\bar{u}_2$ is the diagonal matrix

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Some examples will now be given to illustrate these concepts, after which a method for finding characteristic vectors and characteristic values will be discussed.

EXAMPLE 1 Geometric methods can be used to find characteristic vectors and characteristic values for rotations, reflections, and projections. These make use of the fact that a nonzero vector $\bar{v}$ is a characteristic vector for T if and only if $T\bar{v}$ is a multiple of $\bar{v}$.

Suppose T is a counterclockwise rotation in R^2 through the angle θ and that θ is *not* a multiple of π. If $\bar{v} \neq \bar{0}$, $T\bar{v}$ cannot be a multiple of $\bar{v}$, as shown in Figure 1.

Figure 1

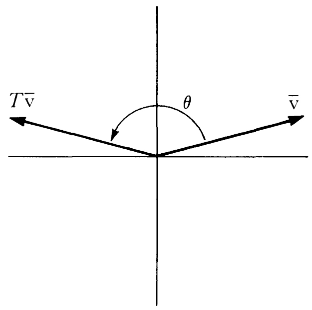
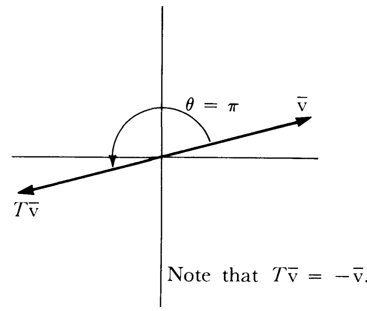


Figure 2



This fact establishes that

> A rotation through θ has no characteristic vectors, if θ is not a multiple of π.

Suppose $\theta = \pi$. Then, as Figure 2 indicates, for *every* nonzero vector $\bar{v}$ we have $T\bar{v} = -\bar{v}$. Since this is also true if θ is any odd multiple of π, we know that

> If θ is an odd multiple of π, then every nonzero vector is a characteristic vector belonging to the characteristic value $\lambda = -1$.

If θ is an even multiple of π, then T is just the identity operator. Thus for every nonzero vector $\bar{v}$ we have $T\bar{v} = \bar{v}$. In other words,

> If θ is an even multiple of π then every nonzero vector is a characteristic vector belonging to the characteristic value $\lambda = 1$.

EXAMPLE 2 Suppose T is the projection in R^2 onto the nonzero vector $\bar{w}$. A vector parallel to $\bar{w}$ is left fixed by T; that is, if $\bar{v}$ is a multiple of $\bar{w}$, $T\bar{v} = \bar{v}$. This shows that every nonzero multiple of $\bar{w}$ is a characteristic vector belonging to the characteristic value $\lambda = 1$.

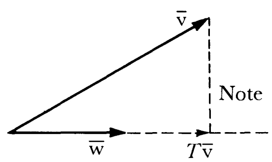
If $\bar{v}$ is orthogonal to $\bar{w}$, then $T\bar{v} = \bar{0}$. Since $\bar{0} = 0\bar{v}$, this statement shows that every nonzero vector $\bar{v}$ which is orthogonal to $\bar{w}$ is a characteristic vector belonging to the characteristic value $\lambda = 0$.

If $\bar{v}$ is neither parallel nor orthogonal to $\bar{w}$, then, as Figure 3 indicates, $T\bar{v}$ is *not* parallel to $\bar{v}$, and such a vector $\bar{v}$ cannot be a characteristic vector.

In summary, we have shown that

> The projection T onto $\bar{w}$ has the characteristic values $\lambda = 0$ and $\lambda = 1$. The nonzero multiples of $\bar{w}$ are the characteristic vectors belonging to $\lambda = 1$, and the nonzero vectors orthogonal to $\bar{w}$ are the characteristic vectors belonging to $\lambda = 0$.

Figure 3



DISCUSSION We extend our terminology to matrices by saying that a vector $\bar{v}$ (written as a column matrix) is a **characteristic vector** for A belonging to the **characteristic value** λ if

$$A\bar{v} = \lambda\bar{v} \qquad \text{and} \qquad \bar{v} \neq \bar{0}$$

Since $I\bar{v} = \bar{v}$, we can rewrite the equation $A\bar{v} = \lambda\bar{v}$ as

$$(\lambda I - A)\bar{v} = \bar{0}$$

Therefore, if $\bar{v}$ is a characteristic vector for A belonging to λ, it follows that $\bar{v}$ is a nonzero solution to $(\lambda I - A)\bar{u} = \bar{0}$; that is, $\bar{v}$ is a nonzero vector in the null space of $\lambda I - A$. Such a $\bar{v}$ can exist only if $\lambda I - A$ is not invertible (Theorem **11**, page 188). We therefore have established that

4 If $\bar{v}$ is a characteristic vector for A belonging to λ, $\lambda I - A$ is not invertible.

If $\lambda I - A$ is not invertible and $\bar{v}$ is a nonzero vector such that $(\lambda I - A)\bar{v} = \bar{0}$, we can then rewrite this equation to conclude that $A\bar{v} = \lambda\bar{v}$ and $\bar{v} \neq \bar{0}$. This shows that

5 If $\lambda I - A$ is not invertible, any nonzero vector $\bar{v}$ in the null space of $\lambda I - A$ is a characteristic vector belonging to the characteristic value λ.

We know from Theorem **14** that $\lambda I - A$ is not invertible if and only if $\det(\lambda I - A) = 0$. The function

$$f(\lambda) = \det(\lambda I - A)$$

is called the **characteristic polynomial** of A. In summary:

THEOREM 15 The characteristic values of A are the roots of the characteristic polynomial $f(\lambda) = \det(\lambda I - A)$. If λ is a root of this polynomial, then any nonzero vector in the null space of $\lambda I - A$ is a characteristic vector belonging to λ.

We have previously noted that f is indeed a polynomial whose degree is the size of A. (See Example **4**, page 222.)

EXAMPLE 3 Theorem **15** provides a method for finding the characteristic values and corresponding characteristic vectors for a matrix. For example, suppose

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

Form the matrix

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 1 \\ -2 & \lambda \end{bmatrix}$$

and take its determinant to obtain the characteristic polynomial of A:

$$\begin{aligned} f(\lambda) &= \det(\lambda I - A) \\ &= (\lambda - 3)\lambda + 2 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 2)(\lambda - 1) \end{aligned}$$

Since the roots of $f(\lambda)$ are $\lambda = 2$ and $\lambda = 1$, Theorem **15** tells us that these are the characteristic values of A. We then find the corresponding characteristic vectors by finding the nonzero vectors in the null spaces of $2I - A$ and $1I - A$. We have

$$2I - A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$1I - A = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

which reduce, respectively, to

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

We see that

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are respective bases for the null spaces of $2I - A$ and $1I - A$. Therefore,

> The characteristic values of A are $\lambda = 2$ and $\lambda = 1$. The nonzero multiples of $\bar{v}_1$ are the characteristic vectors belonging to $\lambda = 2$, while the nonzero multiples of $\bar{v}_2$ are the characteristic vectors belonging to $\lambda = 1$.

This information will be used in the next section to show that A is similar to

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE 4 In using Theorem **15** it is necessary to calculate $\det(\lambda I - A)$. This can be done using the definition of determinant. (See formula **2**, page 217.) The formulas of Exercise **10**, page 230, are, however, worth remembering:

6 If $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $\det(\lambda I - A) = \lambda^2 - (a + d)\lambda + ad - bc$

If $$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \text{ then}$$

7 $$\det(\lambda I - A) = \lambda^3 - (A_{11} + A_{22} + A_{33})\lambda^2 + \left\{\det\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \det\begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix} + \det\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\right\}\lambda - \det A$$

For example, if

$$A = \begin{bmatrix} 8 & 9 & 9 \\ 3 & 2 & 3 \\ -9 & -9 & -10 \end{bmatrix}$$

formula **7** gives

$$\det(\lambda I - A) = \lambda^3 - (8 + 2 - 10)\lambda^2 + \left\{\det\begin{bmatrix} 8 & 9 \\ 3 & 2 \end{bmatrix} + \det\begin{bmatrix} 8 & 9 \\ -9 & -10 \end{bmatrix} + \det\begin{bmatrix} 2 & 3 \\ -9 & -10 \end{bmatrix}\right\}\lambda - \det A$$

Calculation of these determinants gives

$$\det(\lambda I - A) = \lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2)$$

Theorem **15** then tells us that the characteristic values of A are $\lambda = -1$ and $\lambda = 2$. Corresponding characteristic vectors are found by finding the nonzero vectors in the null spaces of $-1I - A$ and $2I - A$.

EXAMPLE 5 The characteristic values of an upper (or lower) triangular matrix are easy to find. For example, if

$$A = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

then

$$f(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 0 & -1 & -2 \\ 0 & \lambda - 2 & 1 & -3 \\ 0 & 0 & \lambda + 3 & -1 \\ 0 & 0 & 0 & \lambda - 4 \end{bmatrix}$$

The determinant of an upper triangular matrix is the product of the diagonal entries. (See property **4c**, page 218.) Therefore

$$f(\lambda) = (\lambda - 2)(\lambda - 2)(\lambda + 3)(\lambda - 4)$$

so that the characteristic values are 2, -3, and 4.

In general,

8 The characteristic values of an upper (or lower) triangular matrix are the diagonal entries of the matrix.

EXERCISES

1 For each of the following operators T describe the characteristic values and vectors. Figures may be helpful.
- **a** T is reflection in R^2 through the line through $\overline{w}$.
- **b** T is projection in R^2 *orthogonal* to $\overline{w}$.
- **c** T is reflection in R^3 through the line through $\overline{w}$.
- **d** T is projection in R^3 onto $\overline{w}$.
- **e** T is projection in R^3 orthogonal to $\overline{w}$.
- **f** T is counterclockwise rotation in R^2 through $\pi/4$ followed by reflection in the x-axis.

2 Suppose $Df = f'$. Show that every real number is a characteristic value of D. [Hint: Calculate $D(e^{ax})$.]

3 Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are characteristic vectors for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

What are the corresponding characteristic values?

4 Use Theorem **15** and formulas **6**, **7**, or **8** to find the characteristic polynomial and the characteristic values for each of the following matrices.

a $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d $\begin{bmatrix} 5 & 1 & 1 \\ -3 & 1 & -3 \\ -2 & -2 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

5 For each of the matrices of Exercise **4** find at least one characteristic vector for each characteristic value.

6 Show that $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ has no real characteristic values. (See also Exercise **11**.)

7 **a** Show that if $A\bar{v} = \lambda\bar{v}$, then $A^2\bar{v} = \lambda^2\bar{v}$.

b Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the characteristic values of A. What are the characteristic values of A^2?

c Suppose

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

What are the characteristic values of A^2? of A^3?

8 **a** Show that if $A\bar{v} = \lambda\bar{v}$, then

$$(A^3 - 3A^2 + A - 2I)\bar{v} = (\lambda^3 - 3\lambda^2 + \lambda - 2)\bar{v}$$

b Suppose $q(\lambda)$ is a polynomial and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the characteristic values of A. What are the characteristic values of $q(A)$? (Hint: See part **a**.)

9 **a** Show that similar matrices have the same characteristic polynomial. [Hint: $\lambda I - P^{-1}AP = P^{-1}(\lambda I - A)P$.]

b How would you define the characteristic polynomial of a linear transformation T from V into V? Does your definition depend upon the choice of basis for V?

10 Suppose A is similar to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

What are the characteristic values of A? (Hint: See Exercise **9a.**)

□ **11** Show that $\begin{bmatrix} 2 \\ 1 - i \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 + i \end{bmatrix}$ are complex characteristic vectors for $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$. What are the corresponding characteristic values? Compare this with Exercise **6**.

□ **12** Find the characteristic polynomial, the complex characteristic values, and at least one corresponding characteristic vector for each of the following.

a $\begin{bmatrix} 2 - i & 2 \\ 1 + 3i & -1 + 3i \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

□ **13** An important theorem, known as the Cayley-Hamilton Theorem, asserts that if $f(\lambda)$ is the characteristic polynomial of A, $f(A)$ is the zero matrix. Verify that this is so for

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$