DETERMINANTS

In Chapter 7, we will encounter n linear algebraic equations in n unknowns. Such equations are best solved by means of determinants, which we discuss in this MathChapter. Consider the pair of linear algebraic equations

$$\begin{aligned} a_{11}x + a_{12}y &= d_1 \\ a_{21}x + a_{22}y &= d_2 \end{aligned} \tag{E.1}$$

If we multiply the first of these equations by a_{22} and the second by a_{12} and then subtract, we obtain

$$(a_{11}a_{22} - a_{12}a_{21})x = d_1a_{22} - d_2a_{12}$$

or

$$x = \frac{a_{22}d_1 - a_{12}d_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{E.2}$$

Similarly, if we multiply the first by a_{21} and the second by a_{11} and then subtract, we get

$$y = \frac{a_{11}d_2 - a_{21}d_1}{a_{11}a_{22} - a_{12}a_{21}} \tag{E.3}$$

Notice that the denominators in both Equations E.2 and E.3 are the same. We represent $a_{11}a_{22} - a_{12}a_{21}$ by the quantity $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, which equals $a_{11}a_{22} - a_{12}a_{21}$ and is called a 2×2 *determinant*. The reason for introducing this notation is that it readily generalizes to the treatment of n linear algebraic equations in n unknowns. Generally, a $n \times n$

determinant is a square array of n^2 elements arranged in n rows and n columns. A 3×3 determinant is given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} \\ -a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{23}a_{32} \end{matrix} \tag{E.4}$$

(We will prove this soon.) Notice that the element a_{ij} occurs at the intersection of the ith row and the jth column.

Equation E.4 and the corresponding equations for evaluating higher-order determinants can be obtained in a systematic manner. First we define a cofactor. The *cofactor*, A_{ij}, of an element a_{ij} is a $(n-1) \times (n-1)$ determinant obtained by deleting the ith row and the jth column, multiplied by $(-1)^{i+j}$. For example, A_{12}, the cofactor of element a_{12} of

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

EXAMPLE E–1

Evaluate the cofactor of each of the first-row elements in

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & -2 & 1 \end{vmatrix}$$

SOLUTION: The cofactor of a_{11} is

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} = 3 - 2 = 1$$

The cofactor of a_{12} is

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = -2$$

and the cofactor of a_{13} is

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} = -6$$

We can use cofactors to evaluate determinants. The value of the 3×3 determinant in Equation E.4 can be obtained from the formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \tag{E.5}$$

Thus, the value of D in Example E–1 is

$$D = (2)(1) + (-1)(-2) + (1)(-6) = -2$$

EXAMPLE E–2

Evaluate D in Example E–1 by expanding in terms of the first column of elements instead of the first row.

SOLUTION: We will use the formula

$$D = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

The various cofactors are

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} = 1$$

$$A_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -1$$

and

$$A_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

and so

$$D = (2)(1) + (0)(-1) + (2)(-2) = -2$$

Notice that we obtained the same answer for D as we did for Example E–1. This result illustrates the general fact that a determinant may be evaluated by expanding in terms of the cofactors of the elements of any row or any column. If we choose the second row of D, then we obtain

$$D = (0)(-1)^3 \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + (-1)(-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = -2$$

Although we have discussed only 3×3 determinants, the procedure is readily extended to determinants of any order.

EXAMPLE E–3

In Chapter 10 we will meet the *determinantal equation*

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} = 0$$

Expand this determinantal equation into a quartic equation for x.

SOLUTION: Expand about the first row of elements to obtain

$$x\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{vmatrix} = 0$$

Now expand about the first column of each of the 3×3 determinants to obtain

$$(x)(x)\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - (x)(1)\begin{vmatrix} 1 & 0 \\ 1 & x \end{vmatrix} - (1)\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 0$$

or

$$x^2(x^2 - 1) - x(x) - (1)(x^2 - 1) = 0$$

or

$$x^4 - 3x^2 + 1 = 0$$

Note that although we can choose any row or column to expand the determinant, it is easiest to take the one with the most zeroes.

A number of properties of determinants are useful to know:

1. The value of a determinant is unchanged if the rows are made into columns in the same order; in other words, first row becomes first column, second row becomes second column, and so on. For example,

$$\begin{vmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 5 & -1 & 2 \end{vmatrix}$$

2. If any two rows or columns are the same, the value of the determinant is zero. For example,

$$\begin{vmatrix} 4 & 2 & 4 \\ -1 & 0 & -1 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

3. If any two rows or columns are interchanged, the sign of the determinant is changed. For example,

$$\begin{vmatrix} 3 & 1 & -1 \\ -6 & 4 & 5 \\ 1 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -1 \\ 4 & -6 & 5 \\ 2 & 1 & 2 \end{vmatrix}$$

4. If every element in a row or column is multiplied by a factor k, the value of the determinant is multiplied by k. For example,

$$\begin{vmatrix} 6 & 8 \\ -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$$

5. If any row or column is written as the sum or difference of two or more terms, the determinant can be written as the sum or difference of two or more determinants according to

$$\begin{vmatrix} a_{11} \pm a'_{11} & a_{12} & a_{13} \\ a_{21} \pm a'_{21} & a_{22} & a_{23} \\ a_{31} \pm a'_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \pm \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix}$$

For example,

$$\begin{vmatrix} 3 & 3 \\ 2 & 6 \end{vmatrix} = \begin{vmatrix} 2+1 & 3 \\ -2+4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

6. The value of a determinant is unchanged if one row or column is added or subtracted to another, as in

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + a_{12} & a_{12} & a_{13} \\ a_{21} + a_{22} & a_{22} & a_{23} \\ a_{31} + a_{32} & a_{32} & a_{33} \end{vmatrix}$$

For example

$$\begin{vmatrix} 1 & -1 & 3 \\ 4 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 0 & 2 \\ 7 & 2 & 3 \end{vmatrix}$$

In the first case we add column 2 to column 1, and in the second case we added row 2 to row 3. This procedure may be repeated n times to obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + na_{12} & a_{12} & a_{13} \\ a_{21} + na_{22} & a_{22} & a_{23} \\ a_{31} + na_{32} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(E.6)}$$

This result is easy to prove:

$$\begin{vmatrix} a_{11}+na_{12} & a_{12} & a_{13} \\ a_{21}+na_{22} & a_{22} & a_{23} \\ a_{31}+na_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + n \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + 0$$

where we used rule 5 to write the first line. The second determinant on the right side equals zero because two columns are the same (rule 2).

We provided these rules because simultaneous linear algebraic equations can be solved in terms of determinants. For simplicity, we will consider only a pair of equations but the final result is easy to generalize. Consider the two equations

$$\begin{aligned} a_{11}x + a_{12}y &= d_1 \\ a_{21}x + a_{22}y &= d_2 \end{aligned} \tag{E.7}$$

If $d_1 = d_2 = 0$, the equations are said to be *homogeneous*. Otherwise, they are called *inhomogeneous*. Let's assume at first that they are inhomogeneous. The determinant of the coefficients of x and y is

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

According to Rule 4,

$$\begin{vmatrix} a_{11}x & a_{12} \\ a_{21}x & a_{22} \end{vmatrix} = xD$$

Furthermore, according to Rule 6,

$$\begin{vmatrix} a_{11}x + a_{12}y & a_{12} \\ a_{21}x + a_{22}y & a_{22} \end{vmatrix} = xD \tag{E.8}$$

If we substitute Equation E.7 into Equation E.8, then we have

$$\begin{vmatrix} d_1 & a_{12} \\ d_2 & a_{22} \end{vmatrix} = xD$$

Solving for x gives

$$x = \frac{\begin{vmatrix} d_1 & a_{12} \\ d_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{E.9}$$

Similarly, we get

$$y = \frac{\begin{vmatrix} a_{11} & d_1 \\ a_{21} & d_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{E.10}$$

Notice that Equations E.9 and E.10 are identical to Equations E.2 and E.3. The solution for x and y in terms of determinants is called *Cramer's rule*. Note that the determinant in the numerator is obtained by replacing the column in D that is associated with the unknown quantity with the column associated with the right sides of Equation E.7. This result is readily extended to more than two simultaneous equations.

EXAMPLE E–4

Solve the equations

$$x + y + z = 2$$
$$2x - y - z = 1$$
$$x + 2y - z = -3$$

SOLUTION: The extension of Equations E.9 and E.10 is

$$x = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ -3 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{9}{9} = 1$$

Similarly,

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{-9}{9} = -1$$

and

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -1 \end{vmatrix}} = \frac{18}{9} = 2$$

What happens if $d_1 = d_2 = 0$ in Equation E.7? In that case, we find that $x = y = 0$, which is an obvious solution called a *trivial solution*. The only way that we could obtain a nontrivial solution for a set of homogeneous equations is for the denominator in Equations E.9 and E.10 to be zero, or for

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{E.11}$$

In Chapter 10, when we discuss ethene, we will meet the equations

$$c_1(\alpha - E) + c_2\beta = 0$$

and

$$c_1\beta + c_2(\alpha - E) = 0$$

where c_1 and c_2 are the unknowns (corresponding to x and y in Equation E.7), α and β are known quantities, and E is the energy of the π electrons. We can use Equation E.11 to derive an expression for the π-electron energies in ethene. Equation E.11 says that for a nontrivial solution (c_1, c_2) to exist, we must have that

$$\begin{vmatrix} \alpha - E & \beta \\ \beta & \alpha - E \end{vmatrix} = 0$$

or that $(\alpha - E)^2 - \beta^2 = 0$. Taking the square root of both sides and solving for E gives

$$E = \alpha \pm \beta$$

Although we considered only two simultaneous homogeneous algebraic equations, Equation E.11 is readily extended to any number. We will use this result in the next chapter.

Problems

E-1. Evaluate the determinant

$$D = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

Add column 2 to column 1 to get

$$\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

and evaluate it. Compare your result with the value of D. Now add row 2 of D to row 1 of D to get

$$\begin{vmatrix} 1 & 4 & 3 \\ -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

and evaluate it. Compare your result with the value of D above.

E-2. Interchange columns 1 and 3 in D in Problem E–1 and evaluate the resulting determinant. Compare your result with the value of D. Interchange rows 1 and 2 of D and do the same.

E-3. Evaluate the determinant

$$D = \begin{vmatrix} 1 & 6 & 1 \\ -2 & 4 & -2 \\ 1 & -3 & 1 \end{vmatrix}$$

Can you determine its value by inspection? What about

$$D = \begin{vmatrix} 2 & 6 & 1 \\ -4 & 4 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

E-4. Find the values of x that satisfy the following determinantal equation

$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 0 & 0 \\ 1 & 0 & x & 0 \\ 1 & 0 & 0 & x \end{vmatrix} = 0$$

E-5. Find the values of x that satisfy the following determinantal equation

$$\begin{vmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{vmatrix} = 0$$

E-6. Show that

$$\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

E-7. Solve the following set of equations using Cramer's rule

$$x + y = 2$$

$$3x - 2y = 5$$

E-8. Solve the following set of equations using Cramer's rule

$$x + 2y + 3z = -5$$

$$-x - 3y + z = -14$$

$$2x + y + z = 1$$

MATHCHAPTER F

MATRICES

Many physical operations such as magnification, rotation, and reflection through a plane can be represented mathematically by quantities called matrices. A matrix is simply a two-dimensional array that obeys a certain set of rules called matrix algebra. Even if matrices are entirely new to you, they are so convenient that learning some of their simpler properties is worthwhile.

Consider the lower of the two vectors shown in Figure F.1. The x and y components of the vector are given by $x_1 = r\cos\alpha$ and $y_1 = r\sin\alpha$, where r is the length of $\mathbf{r}_1$. Now let's rotate the vector counterclockwise through an angle θ, so that $x_2 = r\cos(\alpha+\theta)$ and $y_2 = r\sin(\alpha+\theta)$ (see Figure F.1). Using trigonometric formulas, we can write

$$x_2 = r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$y_2 = r\sin(\alpha+\theta) = r\cos\alpha\sin\theta + r\sin\alpha\cos\theta$$

or

$$\begin{aligned} x_2 &= x_1\cos\theta - y_1\sin\theta \\ y_2 &= x_1\sin\theta + y_1\cos\theta \end{aligned} \tag{F.1}$$

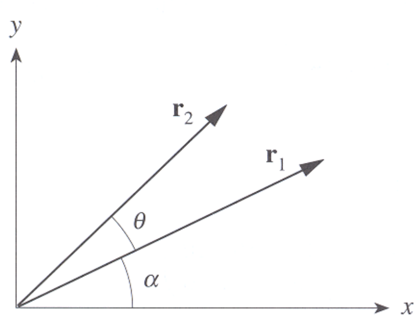

FIGURE F.1
An illustration of the rotation of a vector $\mathbf{r}_1$ through an angle θ.

We can display the set of coefficients of x_1 and y_1 in the form

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{F.2}$$

We have expressed R in the form of a *matrix*, which is an array of numbers (or functions in this case) that obey a certain set of rules, called matrix algebra. We will denote a matrix by a sans serif symbol, e.g., A, B, etc. Unlike determinants (MathChapter E), matrices do not have to be square arrays. Furthermore, unlike determinants, matrices cannot be reduced to a single number. The matrix R in Equation F.2 corresponds to a rotation through an angle θ.

The entries in a matrix A are called its *matrix elements* and are denoted by a_{ij}, where, as in the case of determinants, i designates the row and j designates the column. Two matrices, A and B, are equal if and only if they are of the same dimension and $a_{ij} = b_{ij}$ for all i and j. In other words, equal matrices are identical. Matrices can be added or subtracted only if they have the same number of rows and columns, in which case the elements of the resultant matrix are given by $a_{ij} + b_{ij}$. Thus, if

$$\mathsf{A} = \begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix}$$

then

$$\mathsf{C} = \mathsf{A} + \mathsf{B} = \begin{pmatrix} -1 & 7 & 5 \\ -5 & 4 & 5 \end{pmatrix}$$

If we write

$$\mathsf{A} + \mathsf{A} = 2\mathsf{A} = \begin{pmatrix} -6 & 12 & 8 \\ 2 & 0 & 4 \end{pmatrix}$$

we see that scalar multiplication of a matrix means that each element is multiplied by the scalar. Thus,

$$c\mathsf{M} = \begin{pmatrix} cM_{11} & cM_{12} \\ cM_{21} & cM_{22} \end{pmatrix} \tag{F.3}$$

EXAMPLE F–1

Using the matrices A and B above, form the matrix D = 3A − 2B.

SOLUTION:

$$\mathsf{D} = 3\begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 18 & 12 \\ 3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 2 & 2 \\ -12 & 8 & 6 \end{pmatrix} = \begin{pmatrix} -13 & 16 & 10 \\ 15 & -8 & 0 \end{pmatrix}$$

One of the most important aspects of matrices is matrix multiplication. For simplicity, we will discuss the multiplication of square matrices first. Consider some linear transformations of (x_1, y_1) into (x_2, y_2):

$$\begin{aligned} x_2 &= a_{11}x_1 + a_{12}y_1 \\ y_2 &= a_{21}x_1 + a_{22}y_1 \end{aligned} \tag{F.4}$$

represented by the matrix

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{F.5}$$

Now let's transform (x_2, y_2) into (x_3, y_3):

$$\begin{aligned} x_3 &= b_{11}x_2 + b_{12}y_2 \\ y_3 &= b_{21}x_2 + b_{22}y_2 \end{aligned} \tag{F.6}$$

represented by the matrix

$$\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{F.7}$$

Let the transformation of (x_1, y_1) directly into (x_3, y_3) be given by

$$\begin{aligned} x_3 &= c_{11}x_1 + c_{12}y_1 \\ y_3 &= c_{21}x_1 + c_{22}y_1 \end{aligned} \tag{F.8}$$

represented by the matrix

$$\mathsf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \tag{F.9}$$

Symbolically, we can write that

$$\mathsf{C} = \mathsf{BA}$$

because C results from transforming from (x_1, y_1) to (x_2, y_2) by means of A followed by transforming (x_2, y_2) to (x_3, y_3) by means of B. Let's find the relation between the elements of C and those of A and B. Substitute Equations F.4 into F.6 to obtain

$$\begin{aligned} x_3 &= b_{11}(a_{11}x_1 + a_{12}y_1) + b_{12}(a_{21}x_1 + a_{22}y_1) \\ y_3 &= b_{21}(a_{11}x_1 + a_{12}y_1) + b_{22}(a_{21}x_1 + a_{22}y_1) \end{aligned} \tag{F.10}$$

or

$$\begin{aligned} x_3 &= (b_{11}a_{11} + b_{12}a_{21})x_1 + (b_{11}a_{12} + b_{12}a_{22})y_1 \\ y_3 &= (b_{21}a_{11} + b_{22}a_{21})x_1 + (b_{21}a_{12} + b_{22}a_{22})y_1 \end{aligned}$$

Thus, we see that

$$C = BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} \quad \text{(F.11)}$$

This result may look complicated, but it has a nice pattern which we will illustrate two ways. Mathematically, the ijth element of C is given by the formula

$$c_{ij} = \sum_k b_{ik}a_{kj} \quad \text{(F.12)}$$

For example,

$$c_{11} = \sum_k b_{1k}a_{k1} = b_{11}a_{11} + b_{12}a_{21}$$

as in Equation F.11. A more pictorial way is to notice that any element in C can be obtained by multiplying elements in any row in B by the corresponding elements in any column in A, adding them, and then placing them in C where the row and column intersect. For example, c_{11} is obtained by multiplying the elements of row 1 of B with the elements of column 1 of A, or by the scheme

$$\begin{matrix} \\ \rightarrow \\ \\ \end{matrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{matrix} \downarrow & \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{matrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

and c_{12} by

$$\begin{matrix} \\ \rightarrow \\ \\ \end{matrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{matrix} & \downarrow \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{matrix} = \begin{pmatrix} \cdot & b_{11}a_{12} + b_{12}a_{22} \\ \cdot & \cdot \end{pmatrix}$$

EXAMPLE F–2

Find C = BA if

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

SOLUTION:

$$C = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3+2+1 & 0+8+1 & -1+0+1 \\ -9+0-1 & 0+0-1 & -3+0-1 \\ 3-1+2 & 0-4+2 & 1+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 9 & 0 \\ -10 & -1 & -4 \\ 4 & -2 & 3 \end{pmatrix}$$

EXAMPLE F–3

The matrix R given by Equation F.2 represents a rotation through the angle θ. Show that R^2 represents a rotation through an angle 2θ.

SOLUTION:

$$\mathrm{R}^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

Using standard trigonometric identities, we get

$$\mathrm{R}^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

which represents rotation through an angle 2θ.

Matrices do not have to be square to be multiplied together, but either Equation F.11 or the pictorial method illustrated above suggests that the number of columns of B must be equal to the number of rows of A. When this is so, A and B are said to be *compatible*. For example, Equations F.4 can be written in matrix form as

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{F.13}$$

An important aspect of matrix multiplication is that BA does not necessarily equal AB. For example, if

$$\mathrm{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\mathrm{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathrm{BA} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so $\mathrm{AB} = -\mathrm{BA}$ in this case. If it does happen that $\mathrm{AB} = \mathrm{BA}$, then A and B are said to *commute*.

EXAMPLE F–4

Do the matrices A and B commute if

$$\mathrm{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

SOLUTION:

$$AB = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

so they do not commute.

Another property of matrix multiplication that differs from ordinary scalar multiplication is that the equation

$$AB = O$$

where O is the zero matrix (all elements equal to zero) does not imply that A or B necessarily is a zero matrix. For example,

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A linear transformation that leaves (x_1, y_1) unaltered is called the identity transformation, and the corresponding matrix is called the *identity matrix* or the *unit matrix*. All the elements of the identity matrix are equal to zero, except those along the diagonal, which equal one:

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The elements of I are δ_{ij}, the Kronecker delta, which equals one when $i = j$ and zero when $i \neq j$. The unit matrix has the property that

$$IA = AI \tag{F.14}$$

The unit matrix is an example of a *diagonal matrix*. The only nonzero elements of a diagonal matrix are along its diagonal. Diagonal matrices are necessarily square matrices.

If $BA = AB = I$, then B is said to be the *inverse* of A, and is denoted by A^{-1}. Thus, A^{-1} has the property that

$$AA^{-1} = A^{-1}A = I \tag{F.15}$$

If A represents some transformation, then A^{-1} undoes that transformation and restores the original state. There are recipes for finding the inverse of a matrix, but we won't need them (see Problem F–9, however). Nevertheless, it should be clear on physical grounds that the inverse of R in Equation F.2 is

$$\mathsf{R}^{-1} = \mathsf{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{F.16}$$

which is obtained from R by replacing θ by $-\theta$. In other words, if $\mathsf{R}(\theta)$ represents a rotation through an angle θ, then $\mathsf{R}^{-1} = \mathsf{R}(-\theta)$ and represents the reverse rotation. It is easy to show that R and R^{-1} satsify Equation F.15. Using Equations F.2 and F.16, we have

$$\begin{aligned} \mathsf{R}^{-1}\mathsf{R} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathsf{R}\mathsf{R}^{-1} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

We can associate a determinant with a square matrix by writing

$$\det \mathsf{A} = |\mathsf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

Thus, the determinant of R is

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

and $\det \mathsf{R}^{-1} = 1$ also. If $\det \mathsf{A} = 0$, then A is said to be a *singular matrix*. Singular matrices do not have inverses.

A quantity that arises in group theory, which we will study in the next chapter, is the sum of the diagonal elements of a matrix, called the *trace* of the matrix. Thus, the trace of the matrix

$$B = \begin{pmatrix} 1/2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1/2 \end{pmatrix}$$

is 3, which we write as $\mathrm{Tr}\, B = 3$.

Problems

F-1. Given the two matrices

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

form the matrices $C = 2A - 3B$ and $D = 6B - A$.

F-2. Given the three matrices

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad C = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

show that $A^2 + B^2 + C^2 = \frac{3}{4}I$, where I is a unit matrix. Also show that

$$AB - BA = iC$$

$$BC - CB = iA$$

$$CA - AC = iB$$

F-3. Given the matrices

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

show that

$$AB - BA = iC$$

$$BC - CB = iA$$

$$CA - AC = iB$$

and

$$A^2 + B^2 + C^2 = 2I$$

where I is a unit matrix.

F-4. Do you see any similarity between the results of Problems F–2 and F–3 and the commutation relations involving the components of angular momentum?

F-5. A three-dimensional rotation about the z axis can be represented by the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that

$$\det R = |R| = 1$$

Also show that

$$R^{-1} = R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

F-6. The *transpose* of a matrix A, which we denote by $\tilde{A}$, is formed by replacing the first row of A by its first column, its second row by its second column, etc. Show that this procedure is equivalent to the relation $\tilde{a}_{ij} = a_{ji}$. Show that the transpose of the matrix R given in Problem F–5 is

$$\tilde{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $\tilde{R} = R^{-1}$. When $\tilde{R} = R^{-1}$, the matrix R is said to be *orthogonal.*

F-7. Given the matrices

$$C_3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \sigma_v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_v' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \sigma_v'' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

show that

$$\sigma_v C_3 = \sigma_v'' \qquad C_3 \sigma_v = \sigma_v'$$

$$\sigma_v'' \sigma_v' = C_3 \qquad C_3 \sigma_v'' = \sigma_v$$

Calculate the determinant associated with each matrix. Calculate the trace of each matrix.

F-8. Which of the matrices in Problem F–7 are orthogonal (see Problem F–6)?

F-9. The inverse of a matrix A can be found by using the following procedure:

a. Replace each element of A by its cofactor in the corresponding determinant (see MathChapter E for a definition of a cofactor).
b. Take the transpose of the matrix obtained in step 1.
c. Divide each element of the matrix obtained in Step 2 by the determinant of A.

For example, if

$$\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

then det A = −2 and

$$\mathsf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Show that $\mathsf{AA}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{I}$. Use the above procedure to find the inverse of

$$\mathsf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

F-10. Recall that a singular matrix is one whose determinant is equal to zero. Referring to the procedure in Problem F–9, do you see why a singular matrix has no inverse?

F-11. Consider the simultaneous algebraic equations

$$x + y = 3$$
$$4x - 3y = 5$$

Show that this pair of equations can be written in the matrix form

$$\mathsf{A}\mathbf{x} = \mathbf{c} \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathsf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$$

Now multiply Equation 1 from the left by A^{-1} to obtain

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{c} \tag{2}$$

Now show that

$$\mathsf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -4 & 1 \end{pmatrix}$$

and that

$$\mathbf{x} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

or that $x = 2$ and $y = 1$. Do you see how this procedure generalizes to any number of simultaneous equations?

F-12. Solve the following simultaneous algebraic equations by the matrix inverse method developed in Problem F–11:

$$x + y - z = 1$$

$$2x - 2y + z = 6$$

$$x + 3z = 0$$

First show that

$$\mathsf{A}^{-1} = \frac{1}{13}\begin{pmatrix} 6 & 3 & 1 \\ 5 & -4 & 3 \\ -2 & -1 & 4 \end{pmatrix}$$

and evaluate $\mathbf{x} = \mathsf{A}^{-1}\mathbf{c}$.